Singularities of the continuation of fields and validity of Rayleigh's hypothesis

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To formulate general results concerning the validity of the Rayleigh hypothesis, we first introduce a definition of the foci and antifoci of an analytic curve. Then, we state two lemmas on the properties of an analytic or harmonic function satisfying given conditions on an analytic curve. This allows us to predict the behavior of the analytic continuation of the field in electrostatics. The use of a conformal mapping permits the generalization of this method in electromagnetics and acoustics. As a consequence, we are able to predict the limit of validity of the Rayleigh hypothesis.

I. INTRODUCTION

At the beginning of the century, the Rayleigh method had been the first attempt at solving the problem of diffraction by gratings.[1] This method has been used for many other problems of electromagnetism and acoustics. Rayleigh made an assumption, the so-called Rayleigh hypothesis, which remained unquestioned for almost 50 years, but provoked considerable controversy thereafter. At present, there is no doubt that the Rayleigh hypothesis is neither always valid, nor always invalid. The interested reader may consult recent reviews in this field.[2,3]

However, the controversial aspect of the Rayleigh hypothesis has not died down, due to a second question: in what conditions may the Rayleigh theory be used to determine the field diffracted by a scattering object, even though the Rayleigh hypothesis fails? In this paper, we are not concerned with this second question. Our aim is to establish a mathematical property which allows us to state a very simple and general result concerning the validity of the Rayleigh hypothesis in electromagnetism and acoustics, when the profile of a diffracting object is given by an analytic curve. To this end, we first deal with the Neumann and Dirichlet problem in electrostatics, since it has been shown that the validity of the Rayleigh hypothesis in electromagnetism or acoustics is linked with the properties of the analytical continuation of the field in the corresponding problems of electrostatics.[4]

II. DEFINITION OF THE FOCI AND ANTIFOCI OF AN ANALYTIC CURVE

The notion of foci is well known for conics. Here, we propose a generalization of this notion to analytic curves. Moreover, we introduce the notion of antifoci.

First, let us recall the definition of an analytic curve Γ: let D_t be a domain (open connected set) of the complex t plane and $I \subset D_t$ a real interval. An analytic curve Γ is the image of I through a transformation

$$z = \zeta(t), \tag{1}$$

ζ being a nonconstant analytic function defined in D_t.

Now, if there exists a point $t_0 \in D_t$, such that $\bar{t}_0 \in D_t$, satisfying

$$\zeta'(t_0) = 0, \tag{2}$$

$$\zeta'(\bar{t}_0) \neq 0, \quad \zeta' \text{ being the derivative of } \zeta, \tag{3}$$

the images $z_0 = \zeta(t_0)$ and $\tilde{z}_0 = \zeta(\bar{t}_0)$ of t_0 and $\bar{t}_0$ will be called the associated focus and antifocus of Γ, respectively.

For instance, let us consider the case of a parabola given by the function

$$z = \zeta(t) = 2t + it^2. \tag{4}$$

Its focus z_0 will be obtained by setting

$$\zeta'(t_0) = 2 + 2it_0 = 0, \tag{5}$$

which means that $t_0 = i$, thus

$$z_0 = i, \tag{6}$$

$$\tilde{z}_0 = -3i. \tag{7}$$

Finally, the antifocus is symmetrical to the focus with respect to the directrix of the parabola.

More generally, it can be verified that the notion of focus given here identifies with the classical one in the case of conics (except for a circle!). When the analytic curve Γ is given by the equation

$$G(x,y) = 0, \tag{8}$$

where G is an analytic function of the variables x and y ($z = x + iy$), it can be shown that a focus z_0 is obtained by

$$z_0 = x_1 + iy_1, \tag{9}$$

where x_1 and y_1 are complex numbers satisfying the system

$$G(x,y) = 0, \tag{10}$$

$$\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y} = 0, \quad \text{with } \frac{\partial G}{\partial x} \neq 0. \tag{11}$$

In addition, the associated antifocus is given by

$$\tilde{z}_0 = \bar{x}_1 + i\bar{y}_1. \tag{12}$$

We shall set

$$z_0 = x_0 + iy_0, \quad \tilde{z}_0 = \tilde{x}_0 + i\tilde{y}_0,$$

where x_0, y_0, $\tilde{x}_0$, $\tilde{y}_0$, the Cartesian coordinates of the focus and the antifocus, are real.

We define a focal line as the image $\zeta(L)$ of a curve L (a) joining t_0 to $\bar{t}_0$ in D_t, (b) symmetrical with respect to the real axis, and (c) intersecting I. For example, in the case of a parabola, the segment $[z_0, \tilde{z}_0]$ is a focal line.

A domain D will be called a focal domain if (a)$D \subset D_z = \zeta(D_t)$ and (b) whenever D contains an antifocus $\tilde{z}_0$, it includes an associated focal line.

It is interesting to notice that with the new variables

$$z = x + iy, \tag{13}$$

$$\tilde{z} = x - iy, \tag{14}$$

the focus is given by

$$H(z, \tilde{z}) = 0, \tag{15}$$

$$\frac{\partial H}{\partial \tilde{z}} = 0 \quad \text{and} \quad \frac{\partial H}{\partial z} \neq 0, \tag{16}$$

with

$$H(z, \tilde{z}) = G(x,y).$$

It is worth noting that a system of parametric equations similar to (1) may be deduced from (10) by integrating the system (Hamilton's canonical equations!)

$$\frac{dx}{dt} = -\frac{\partial G}{\partial y}, \tag{17}$$

$$\frac{dy}{dt} = \frac{\partial G}{\partial x}, \tag{18}$$

with arbitrary initial conditions.

With the new variables defined in (13) and (14), these equations become

$$\frac{dz}{dt} = 2i\frac{\partial H}{\partial z}, \tag{19}$$

$$\frac{d\tilde{z}}{dt} = -2i\frac{\partial H}{\partial \tilde{z}}. \tag{20}$$

III. LEMMAS

Lemma 1: An analytic curve Γ being given, let $F(z)$ be an analytic function in a focal domain D and $\tilde{z}_0$ an antifocus in D. If, for $z \in \Gamma \cap D$, $F(z)$ is real, then $F'(\tilde{z}_0) = 0$.

Proof: The function

$$\theta(t) = F(\zeta(t)) \tag{21}$$

is analytic in the connected component of $\zeta^{-1}(D)$ which contains $t_0, \bar{t}_0$. If $t \in I$, $\theta(t)$ is real, thus $\theta'(t)$ is real, too, and therefore, from a well-known symmetry property,

$$\theta'(\bar{t}_0) = \overline{\theta'(t_0)}. \tag{22}$$

But,

$$\theta'(t_0) = \zeta'(t_0)F'(z_0), \tag{23}$$

$$\theta'(\bar{t}_0) = \zeta'(\bar{t}_0)F'(\tilde{z}_0), \tag{24}$$

and from (2) and (3)

$$F'(\tilde{z}_0) = 0. \tag{25}$$

Lemma 2: An analytic curve Γ being given, let $u(x,y)$ be a harmonic function in a focal domain D and $\tilde{z}_0$ an antifocus in D. If, for $z \in \Gamma$, $u(x,y)$ (or its normal derivative) vanishes, then $\tilde{z}_0$ is a saddle point of $u(x,y)$

$$\frac{\partial u}{\partial x}(\tilde{x}_0,\tilde{y}_0) = \frac{\partial u}{\partial y}(\tilde{x}_0,\tilde{y}_0) = 0. \tag{26}$$

Proof: D can be supposed to be simply connected without loss of generality. There exists an analytic function $F(z)$ such that

$$u(x,y) = \mathrm{Im}(F(z)) \quad [\text{or } u(x,y) = \mathrm{Re}(F(z))],$$

where $F(z)$ fulfills the conditions of Lemma 1. Hence $F'(\tilde{z}_0) = 0$, which is equivalent to (26).

IV. EXAMPLES OF APPLICATION

(1) Let D be a domain intersecting an analytic curve Γ and containing an antifocus $\tilde{z}_0$. Let F be analytic in D and real on Γ. Then, an analytic continuation of F cannot be made along a focal line up to the associated focus z_0, unless $F'(\tilde{z}_0) = 0$.

Such an analytic continuation can be deduced from the symmetry property of $F(\zeta(t))$.

(2)Let us consider a Jordan domain Ω with analytic boundary Γ and a conformal mapping $Z = \phi(z)$ of the exterior of Γ on the exterior of the unit circle C (Fig. 1). We have locally $\phi(z) = \exp(iF(z))$, where F is real on Γ. Moreover, F' is analytic and different from 0 outside $\Omega + \Gamma$. This entails that the foci of Γ located in Ω are singularities of the analytic continuation of ϕ along the focal lines.

(3) A third example consists of the homogeneous Dirichlet and Neumann problems for the Laplace equations.

Now, we shall restrict ourselves to the case where Γ separates the space in two complementary regions Ω_1 and Ω_2. These regions are unbounded if Γ goes to infinity, but one of them, Ω_2, is bounded (the interior region) if Γ is a Jordan curve.

We consider a harmonic function $u(x,y)$ defined in Ω_1 and which satisfies a homogeneous Dirichlet or Neumann condition on Γ. If Ω_1 contains an antifocus $\tilde{z}_0$, then the continuation of u across Γ along a focal line will not be possible at the associated focus z_0 if $(\partial u/\partial x)(\tilde{x}_0,\tilde{y}_0) \neq 0$ or $(\partial u/\partial y)(\tilde{x}_0,\tilde{y}_0) \neq 0$.

Indeed, if this continuation were possible, $u(x,y)$ would be harmonic in a focal domain containing z_0 and $\tilde{z}_0$, a fact which entails that the partial derivatives of u with respect to x and y vanish at the point $(\tilde{x}_0,\tilde{y}_0)$.

V. VALIDITY OF SOME EXPANSIONS OF THE FIELD USED IN ELECTROMAGNETICS AND ACOUSTICS

We consider the Helmholtz equation

$$\nabla^2 u(x,y) + k^2 u(x,y) = 0, \quad \text{in } \Omega_1, \tag{27}$$

with the homogeneous Dirichlet or Neumann conditions on Γ [notations of Sec. IV, example (3)].

It has been shown[5] that the use of a conformal mapping $Z = \Phi(z)$ which maps Ω_1 on the upper Z half-plane or on the

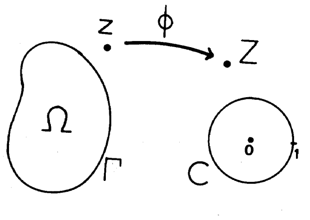

FIG. 1. A property of the conformal mapping.

exterior of the unit disk allows one to define an equivalent problem in the Z complex plane, where $v(X,Y) = u(x,y)$ satisfies the Dirichlet or Neumann boundary conditions on the real axis or the unit circle and a new Helmholtz equation

$$\nabla^2 v(X,Y) + k^2 \left| \frac{dz}{dZ} \right|^2 v(X,Y) = 0. \qquad (28)$$

We have already seen that the continuation of Φ in Ω_2 is singular at a focus z_0 of Γ. This entails that even though $v(X,Y)$ is regular in the upper half-plane, we can expect a singularity of the continuation of $u(x,y)$ in Ω_2 at the focus since dZ/dz is singular at this point. Of course, this rule is not general since we have no information about the value of $v(X,Y)$ at the image of the focus.

It is clear that our criterion gives a means to locate *some* of the singularities of the conformal mapping. Other singularities may well exist in the complementary domain. On the other hand, we emphasize that the criterion does not *guarantee* a singularity at the focus in Ω_2.

The location of the singularity of the analytical continuation of the field in Ω_2 allows one to predict the validity of some expansions of the field used in electromagnetics and acoustics. The most famous of these expansions has been used by Lord Rayleigh to represent the field diffracted by a grating.[1] The reader interested in the study of the validity of Rayleigh's hypothesis may refer to recent reviews in this field (for instance, see Ref. 3 and included references).

Here, we first deal with the more general case where Γ is a modulated two-dimensional surface extending to infinity (Fig. 2), obtained by deforming a mirror placed on the $0x$ axis. An incident wave u^i propagating in Ω_1 is impinging on Γ. The equivalent of Rayleigh's hypothesis is to assume that in Ω_1, the diffracted field $u^d = u - u^i$ (where u denotes the total field) can be expressed in the form of a sum of plane waves

$$u^d = \int_{-\infty}^{\infty} a(\alpha) \exp(i\alpha x + i\beta y) d\alpha, \qquad (29)$$

with $\beta = \sqrt{k^2 - \alpha^2}$ or $i\sqrt{\alpha^2 - k^2}$, the time dependence of the field being in $\exp(-i\omega t)$.

Let us show briefly the great interest of this kind of representation of the field. Indeed, the right-hand member of Eq. (29) obviously satisfies the Helmholtz equation and the outgoing wave condition at infinity. So, if this representation is valid everywhere above Γ, it can be used to express the

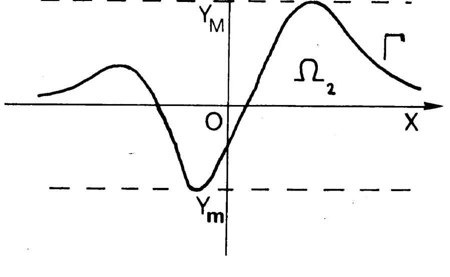

FIG. 2. Validity of the plane wave expansion in the problem of modulated surface.

third condition of the boundary value problem, viz., the boundary condition on Γ. This gives a very simple tool to solve the diffraction problem. It is not so for other rigorous methods which can lead to the solving of integral equations or differential systems of infinite order.

It can be demonstrated that the integral in the right-hand side of (29) actually represents u^d above the top y_M of Γ (the demonstration of this property and those used in the following can be found in Ref. 3 for the particular case of diffraction gratings). Below y_M, the integral is equal to the diffracted field or its analytic continuation in Ω_2, *provided it converges*. Obviously, this integral cannot converge below a focus (except if this focus is not a singularity of the continuation of u). Indeed, since $\exp(i\beta y)$ behaves like $\exp - |\alpha| y$ when $|\alpha| \to \infty$, this integral cannot converge at a point of ordinate y' if it diverges at a point of ordinate $y > y'$.

So, it can be expected that the expansion of u^d given by (29) cannot represent the diffracted field in Ω_1 if a focus is located above the bottom y_M of Γ. This means that a method using this integral to express the boundary condition on Γ fails, at least from a theoretical point of view. Finally we can state the following rule: The plane wave expansion given by the right-hand side of (29) in general cannot represent the diffracted field in Ω_1 when a focus of Γ in Ω_2 is located above the bottom of Γ.

It must be remarked that, in the particular case where Γ is a periodic curve, a profile of a diffraction grating, similar criterion have been given by some authors using conformal mapping[6,7] or the steepest descent method.[8,9]

For instance, let us consider the curve Γ given by

$$y = 2a/\cosh x, \quad \text{with } a > 0, \qquad (30)$$

located above the $0x$ axis.

From Eqs. (10) and (11), we deduce that the foci are given by the equation

$$\sin^2 v + 2a \sin v - 1 = 0, \quad \text{where } v = ix. \qquad (31)$$

There exists an infinity of foci. From the point of view of the validity of the Rayleigh expansion, the most important is

$$z_0 = iy_0, \qquad (32)$$

with

$$y_0 = (2a\sqrt{1+a^2} + 2a^2)^{1/2} - \arcsin(\sqrt{1+a^2} - a). \qquad (33)$$

This focus is located on the imaginary axis ($y_0 \to -\pi/2$ for $a \to 0$) and crosses the real axis for $a = 0.280\,548...$ (the corresponding antifocus being located in Ω_1).

So, we can expect a failure of the plane-wave expansion method for larger values of a. The study of the other foci does not modify this conclusion.

Now, let us consider a second kind of curve: the Jordan curve (Fig. 3). In that case, it can be shown that, if an incident wave propagates in Ω_1, the field outside a circle of radius ρ_M centered on 0 can be represented by a series

$$u^d(P) = \sum_{-\infty}^{\infty} a_n H_n^{(1)}(kr) \exp(in\theta), \qquad (34)$$

a_n being complex coefficients, $H_n^{(1)}$ Hankel functions, and (r,θ) the polar coordinates of a point P.

Considerations similar to those described for modulated surfaces demonstrate the following rule: The expansion given by the right-hand side of (34) in general cannot represent the diffracted field everywhere in Ω_1 when a focus of Γ in Ω_2 is located between the two dotted circles of Fig. 3, of radius ρ_M and ρ_m.

Let us apply this rule to the curve Γ given by

$$x^4 + y^4 = 1. \quad (35)$$

To find the foci of Γ, we use Eqs. (15) and (16), and remarking that (35) becomes

$$H(z,\bar{z}) = \tfrac{1}{32}(z^4 + 6z^2\bar{z}^2 + \bar{z}^4 - 8) = 0, \quad (36)$$

it turns out that $12z^2\bar{z} + 4\bar{z}^3 = 0$, i.e.,

$$\bar{z} = 0, \quad (37)$$

or

$$\bar{z}^2 + 3z^2 = 0. \quad (38)$$

Putting (37) into (36) shows that $z^4 = 8$, and the associated foci are given by

$$z_0 = 2^{3/4}\exp(in(\pi/2)), \quad n = 0,1,2,3. \quad (39)$$

These foci are located in Ω_1 and have no interest for our problem. Now, Eqs. (36) and (38) lead to the equation $z^4 = -1$, which means that the second set of foci is given by

$$z_0 = \exp[i(\pi/4 + n\pi/2)], \quad n = 0,1,2,3. \quad (40)$$

We are led to an amazing conclusion: four foci are just located on the circle of radius $\rho_m = 1$, which means that the expansion (34) actually can represent the field in Ω_1, but diverges just below the points of Γ located on the two axes of coordinate and placed on the circle $r = \rho_m$.

It is worth noting that Eqs. (19) and (20) allow one to find parametric equations associated with Eq. (35), using elliptic functions.

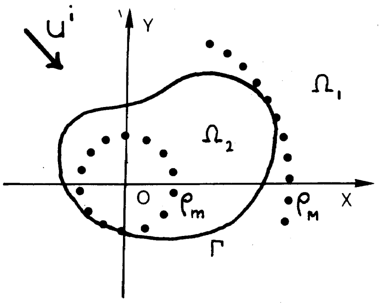

FIG. 3. Validity of a simple representation of the field for a Jordan curve.

VI. CONCLUSION

Introducing the notion of focus and antifocus has allowed us to state in a very simple and general form a property of the singularities of the continuation of the field. As a consequence, we can predict the theoretical limits of some simple expansions used to solve a large class of boundary problems in electromagnetics and acoustics.

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